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Generation of complex bipartite graphs by using a preferential rewiring process

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It is important in computer science, sociology, and so on to investigate complex bipartite graphs from a viewpoint of statistical physics. We propose a model to generate complex bipartite graphs without growing; the bipartite graphs are assumed to have two sets of the fixed numbers of nodes and a fixed number of edges between nodes belonging to different sets of nodes. In this model, essential ingredients are a preferential rewiring process and a fitness distribution function. By using the preferential rewiring process, we confirm that a bipartite graph reaches a stationary state after a sufficiently long time has passed. We find that the obtained bipartite graph has a scale-free-like property when a suitable fitness distribution is used. It turns out that a condensation of edges takes place in the cases of certain fitness distributions.

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I. INTRODUCTION

Many complex systems have networks as their backbone. In the networks, nodes represent elementary units composing a complex system and edges express interactions or relations between pairs of elementary units. In recent studies, a scale-free property has been discovered in the Internet [1], the World Wide Web [2], social networks [3–6], metabolic networks [7,8], and so on. A network with the scale-free property is characterized by the degree distribution $P(k)$ with a power-law behavior; $P(k)$ is defined as the probability that a node is connected to k other nodes. It has been revealed that the scale-free networks have several interesting properties different from regular lattices or random networks proposed by Erdös and Rényi [9], so that big effort has been devoted to the study of complex networks recently [10–12].

Some of the complex networks in the real world have bipartite graphs which are suitable to represent their network structure. A bipartite graph is defined as a network in which nodes are divided into two sets so that no edge connects two nodes in the same set. In the social science literature, the bipartite graph is called a *collaboration network*. The collaboration network is generally defined as follows: there are two sets of nodes, one representing a set of actors, and the other a set of collaboration acts. An edge is a connection between an actor and a collaboration act which is participated by the actor. Therefore, the actors participating in a common collaboration act relate to each other through that act. Figure 1(a) shows an example of bipartite graphs; there are 7 actors and 4 collaboration acts. We see that there are no edges between nodes in the same set, but there exist edges connecting actors and collaboration acts. One of examples of the collaboration networks is a *scientific collaboration network*. In the scientific collaboration network, actors represent scientists, and collaboration acts denote scientific papers. An edge is connected between a scientist and a paper written by the scientist, and scientists are related by their coauthorship of scientific papers. One of the other examples of collaboration networks is a *corporate board and director network* [13], in which actors represent directors in companies and collaboration acts stand for companies. An edge connects a director and a company if the director sits on the board of the company.

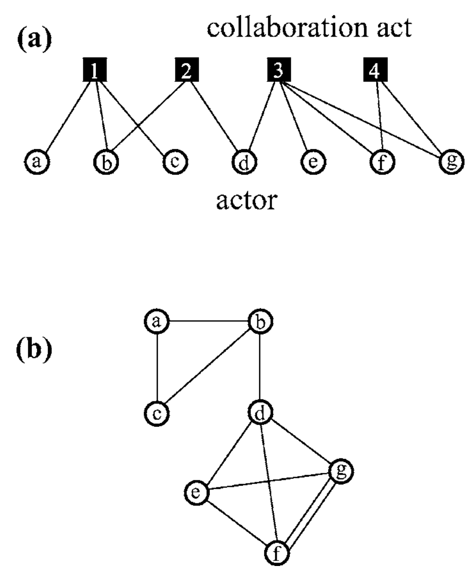

FIG. 1. (a) An example of bipartite graphs. (b) The one-mode projection of the network in (a).

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While it has been revealed that some of the collaboration networks have the scale-free property [3–6], much study has not been done on the generation of complex networks with the scale-free property expressed by bipartite graphs. Newman *et al.* [14] have proposed a model for generating a complex bipartite graph, in which two degree distributions for both types of nodes are given and edges are randomly connected under the constraint that their degree distributions are fixed. Ramasco *et al.* [4] and Ramezanpour [15] have proposed the other type of models for generation of complex bipartite graphs; those are growing networks in which the number of nodes and that of edges are increasing and newly added edges are connected to nodes by a suitable rule. In order to generate a complex network of bipartite graph with the scale-free property, Ramasco *et al.* [4] have applied a rule, which is based on the concept of the preferential attachment proposed by Barabási and Albert [16,17], to their model for generating complex networks.

A certain class of real collaboration networks are described by growing bipartite graphs. The scientific collaboration network is one of such growing networks. In some cases, a scientist has collaboration with other scientists for research activities, and as a result, a paper is written in cooperation with those scientists. Thus, once edges connect a scientist and a paper, those edges in the network are fixed and not removed. However, there is the other types of collaboration networks in which the size of networks does not grow rapidly, but the structure of the networks changes with time rapidly. For instance, in corporate board and director networks, we consider that the number of companies increases slowly and hence is considered to be fixed approximately, but directors in companies change in turn rather rapidly. This is a kind of adiabatic approximation. Therefore, we use a nongrowing bipartite graph for describing the network structure in this case; the nongrowing bipartite graph is defined that the numbers of resources such as actors, collaboration acts, and edges, are fixed, but the edges in the graph are dynamically rewired. Recently, nongrowing models for unipartite graphs, which consist of only one set of nodes, have attracted attention to scientists, and then several models have been proposed in order to generate nongrowing complex networks with the scale-free property [18–21]. A nongrowing model for bipartite graphs has recently been proposed [22], and the existence and the size of the giant component have been investigated.

In the present paper, we propose a model for the complex bipartite graph with the fixed numbers of resources; essential ingredients are a preferential rewiring and a fitness distribution function in the model. We show that the model generates the complex networks with the scale-free property for some fitness distribution functions. The fitness parameters correspond to different abilities to compete for edges, and it seems reasonable that each node (agent) in the real world has a different ability. It is revealed that the randomness of fitness parameters is necessary for a fat-tailed degree distribution, and a non-fat-tailed fitness distribution can generate a network with a fat-tailed degree distribution, as in the case of the threshold model [18]. Furthermore, a condensation phenomenon is also investigated from a viewpoint of statistical physics; the critical exponents are estimated by using the finite size scaling.

The outline of the paper is as follows. In Sec. II, we introduce a notation of a bipartite graph and explain a one-mode projection of the bipartite graph. We introduce a model of generating complex bipartite graphs in Sec. III, and present an analytical treatment for a degree distribution by using a rate equation approach. Section IV contains results of numerical experiments. We show the degree distributions of obtained bipartite graphs, and then compare these degree distributions with those of their one-mode projections. We also make a comparison between the results obtained by the analytical treatment described in Sec. III and those by numerical experiments. An unexpected phenomenon, so-called *condensation of edges*, has been found in the results of numerical experiments in Sec. IV, and hence we investigate the phenomenon in more detail in Sec. V. Finally, we draw the main conclusions in Sec. VI.

II. BIPARTITE GRAPH AND ONE-MODE PROJECTION

A bipartite graph consists of two sets of nodes. In the present paper, we assign each node in one set to a *collaboration act* and each node in the other set to an *actor*. An edge exists only between a collaboration act and an actor, but there is no edge between two nodes in the same set. We denote the number of actors as M and that of collaboration acts as N. A bipartite graph is represented by an $M \times N$ adjacency matrix $B=(b_{i\alpha})_{M\times N}$, whose components are $b_{i\alpha}=1$ if there is an edge between actor i and collaboration act α, and 0 otherwise. Note that the adjacency matrix B is a binary matrix, and is neither square nor symmetric in general.

Next, we consider a one-mode projection of a bipartite graph. The one-mode projection produces a network which is composed of the actors connected to each other whenever they share a collaboration act. We call a network obtained by one-mode projection from a bipartite graph simply as *a one-mode projection* in the present paper. The $M \times M$ adjacency matrix of the one-mode projection, $F=(f_{ij})_{M\times M}$, is defined as

$$f_{ij} = \begin{cases} 0 & \text{if } i=j, \\ \sum_{\alpha=1}^{N} b_{i\alpha} b_{j\alpha} & \text{otherwise}. \end{cases} \tag{1}$$

Here we set the diagonal elements to 0. If actors i and j do not belong to the same collaboration act α, we have either $b_{i\alpha}=0$ or $b_{j\alpha}=0$ (or the both are zero), and hence we calculate the number of collaboration acts connecting actor i and actor j by using the second expression of the right-hand side in Eq. (1). We notice that the adjacency matrix is symmetric.

The one-mode projection is allowed to have multiple edges, and hence we define the degree of actor i in the one-mode projection, k_i, as follows:

$$k_i = \sum_j f_{ij}. \tag{2}$$

We consider the degree distribution of the one-mode projection, $P(k)$, using the degree defined by Eq. (2). We notice here that "one-mode projection" is used as a synonym of "weighted network" and k_i of Eq. (2) is also called as "strength."

Ramasco *et al.* [4] have investigated properties of the ordinary random graphs as for the one-mode projection. In Sec. IV B, we make a comparison between the degree distribution of actors, $Q(m)$, and the one of the one-mode projection, $P(k)$. We also illustrate one-mode projections obtained by our model with several examples in Sec. V.

III. NONGROWING MODEL FOR COMPLEX BIPARTITE GRAPH

A. Preferential rewiring process

We consider an undirected bipartite graph. We assume that the number of collaboration acts, that of actors, and that of edges are fixed, respectively, in the bipartite graph. Two independent degree distributions are defined in the bipartite graph: one is the degree distribution of collaboration acts, $S(n)$, which is the probability that any collaboration act has n actors participating in a collaboration act; the other is the degree distribution of actors, $Q(m)$, which shows the probability that any actor takes part in m collaboration acts. In the present paper, we neglect actors with no edges when we evaluate the degree distribution of actors, $Q(m)$, so that m should be greater than or equal to 1 in the argument of the degree distribution $Q(m)$. Collaboration acts with only one edge do not play a role in connecting actors, and then we have $n \geqslant 2$ in the argument of the degree distribution of collaboration acts, $S(n)$.

We explain a method for generating initial networks. In empirical data on bipartite graphs, it has been reported that some bipartite graphs have an exponential degree distribution of collaboration acts and an exponential or a power-law degree distribution of actors [4]. Therefore, we fix the degree distribution of collaboration acts, $S(n)$, as an exponential form. The degree distribution $S(n)$ is now defined as

$$S(n) = \frac{1}{\langle n\rangle - 2}\exp\left(-\frac{n-2}{\langle n\rangle - 2}\right), \tag{3}$$

where $\langle n\rangle$ is the average degree of the collaboration acts and n is an integral value with $n \geqslant 2$. Then, we focus our attention on the degree distribution of actors, $Q(m)$. We are interested in whether the degree distribution $Q(m)$ becomes an exponential form or a power-law form.

We have four parameters for generating initial networks; those are the number of actors, M, that of collaboration acts, N, the average degree of actors, $\langle m\rangle$, and the average degree of collaboration acts, $\langle n\rangle$. However, there is one relationship among these parameters:

$$N\langle n\rangle = M\langle m\rangle = K, \tag{4}$$

where K is the total number of edges in the bipartite graph. In the present paper, we set initially the number of actors, M, that of collaboration acts, N, and the average degree of actors, $\langle m\rangle$. Using these parameters, we calculate the average degree of collaboration acts, $\langle n\rangle$, by means of Eq. (4).

An initial network is generated as follows.

(I) We generate N random numbers, $\{n_\alpha | 1 \leqslant \alpha \leqslant N\}$, from a given distribution $S(n)$ and assign n_α to collaboration act α. The assigned number, n_α, is the number of edges connected to collaboration act α.

(II) For each collaboration act α, n_α different actors are selected randomly and connected to the collaboration act α. Note that there should be only one edge between a collaboration act and an actor; multiple edges are not allowed.

After preparing an initial network, we apply a preferential rewiring process for the initial network. The preferential rewiring process is the following operations.

(i) We select randomly a collaboration act α and an edge $\hat{l}_{\alpha j}$ which connects collaboration act α to actor j.

(ii) The edge $\hat{l}_{\alpha j}$ is removed, and replace it with a new edge $\hat{l}_{\alpha j'}$ that connects the collaboration act α to an actor j' chosen randomly with a probability

$$\Pi_{j'} = \frac{\eta_{j'}(m_{j'}+1)}{\sum_j \eta_j(m_j+1)}, \tag{5}$$

where m_j is the number of edges connected to actor j. If there is already the edge $\hat{l}_{\alpha j'}$, we choose a different actor j'' with the probability $\Pi_{j''}$. A coefficient η_i is a fitness parameter, which represents that each actor has a different ability to compete for edges. A value of the fitness parameter η_i is chosen from a fitness distribution $\rho(\eta)$. We assume that once the fitness parameter is assigned to each actor, it does not change in time.

Note that the probability Π_i is not proportional to m_i, but to m_i+1, so that there is a nonzero probability that isolated actors ($m_i=0$) acquire new edges.

By using the preferential rewiring process, we find that the network reaches a stationary state in the numerical experiments after sufficiently many operations. Hence we analyze the stationary state after $300M$ operations for each case, if otherwise is not stated, because we have checked stationary states are obtained at near $200M$ operations in those numerical experiments [23].

B. Analytical treatment

Using a rate equation approach, we describe a time evolution of the average number of actors, with which we may calculate a degree distribution of actors, $Q(m)$. The similar approach using a generating function is given in Ref. [21].

We denote the average number of actors with m edges and fitness parameter η in $[\eta, \eta+d\eta]$ at time t as $q_m(\eta,t)$. The time evolution of $q_m(\eta,t)$ is described as

$$\frac{\partial q_m(\eta,t)}{\partial t} = -\frac{(m+1)\eta q_m(\eta,t)}{Z(t)} + \frac{m\eta q_{m-1}(\eta,t)}{Z(t)} - \frac{m q_m(\eta,t)}{M\langle m\rangle} + \frac{(m+1)q_{m+1}(\eta,t)}{M\langle m\rangle}. \tag{6}$$

We show all the processes in the right-hand side of Eq. (6) in Fig. 2. The first term on the right-hand side of Eq. (6) represents the loss of actors with m edges when they obtain a new edge, while the second term is due to the increase in the number of actors with m edges when an actor with $m-1$

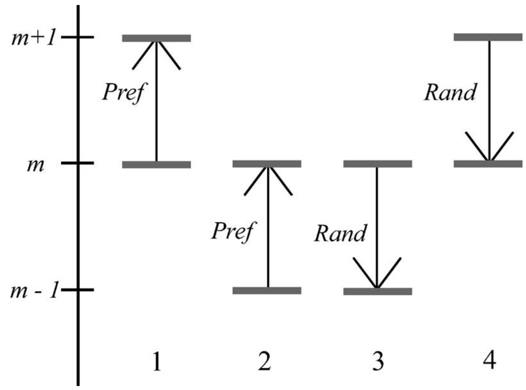

FIG. 2. The processes in the right-hand side of Eq. (6). The process 1 denotes the first term, the process 2 the second, and so on. The arrows show how the number of edges of an actor changes. After a collaboration act and an edge connected to the collaboration act are randomly chosen, we remove the edge which corresponds to the arrows in the processes 3 and 4. In the processes 1 and 2, the arrows correspond to rewiring to preferentially chosen actors.

edges obtains a new edge. The third and last terms are due to the loss of actors with m and $m+1$ edges by losing edges to be rewired, respectively. As stated before, M is the number of actors and $\langle m \rangle$ is the average degree of actors. $Z(t)$ and $M\langle m \rangle$ are normalization factors; the factor $Z(t)$ is defined by $Z(t)=\Sigma_m(m+1)\int \eta q_m(\eta,t)d\eta$.

In order to consider the stationary solution of Eq. (6), we set $q_m(\eta)\equiv q_m(\eta,t\to\infty)$. The stationary solution then satisfies the following equation:

$$\frac{(m+1)\eta}{Z}q_m(\eta) - \frac{m+1}{M\langle m \rangle}q_{m+1}(\eta) = \frac{m\eta}{Z}q_{m-1}(\eta) - \frac{m}{M\langle m \rangle}q_m(\eta) = \text{const}, \tag{7}$$

where $Z=\Sigma_m(m+1)\int \eta q_m(\eta)d\eta$. In the present paper, we set the constant to zero by assuming the detailed balance in the stationary state between the state with $m-1$ edges and that with m edges (or the state with m edges and that with $m+1$ edges). Thus we have the following equation from Eq. (7):

$$\frac{m\eta}{Z}q_{m-1}(\eta) - \frac{m}{M\langle m \rangle}q_m(\eta) = 0. \tag{8}$$

From the single recurrent Eq. (8), we obtain the expression of $q_m(\eta)$ given as follows:

$$q_m(\eta) = q_0(\eta)\left(\frac{\eta M\langle m \rangle}{Z}\right)^m. \tag{9}$$

Note that when $Z<\eta M\langle m \rangle$, the value of $q_m(\eta)$ diverges as the value of m increases, and hence another method will be needed for evaluating the degree distribution. The number of actors with m edges is calculated by

$$q_m = \int d\eta\, \rho(\eta) q_m(\eta). \tag{10}$$

Then, the degree distribution of actors, $Q(m)$, is obtained by

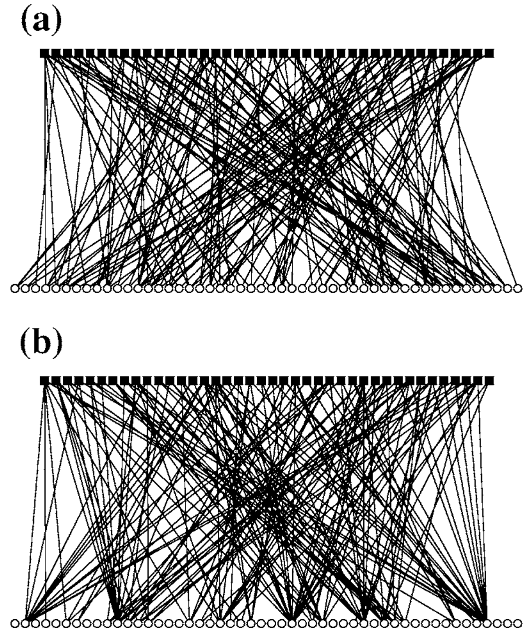

FIG. 3. (a) An example of initial networks generated by the method described in Sec. III A. The network has the following parameter values: $M=50$, $N=40$, $\langle m \rangle=3.0$, and $\langle n \rangle=3.75$. (b) An example of networks obtained by our model after reaching a stationary state. In this example, we have used the uniform fitness distribution $\rho(\eta)=1$ $(0\leq\eta\leq1)$ and repeated $100M$ steps of the preferential rewiring process.

$$Q(m) = \frac{q_m}{\sum_{m=1}^{\infty} q_m}. \tag{11}$$

In Eq. (9), $q_0(\eta)$ and Z are not explicitly known and hence we cannot derive the degree distribution in general. But we can explicitly derive the degree distribution for the case of the delta-function fitness distribution $\rho(\eta)=\delta(\eta-1)$; in this case we just consider only actors with fitness parameter 1 and hence the factor Z is easily calculated. The analytical result in the case of the delta-function fitness distribution is presented in Sec. IV D.

IV. RESULTS OF OBTAINED COMPLEX BIPARTITE GRAPH

A. Basic character of the network obtained by the model

One of the features of networks obtained by our model is that the network has many isolated actors. When we make the one-mode projection of the obtained bipartite graph, we find that there are many isolated actors and only one giant cluster of actors. In other words, there is no small cluster at all, but only one giant cluster exists; a small cluster means an isolated cluster in which a few actors are connected to each other. This feature stems from a "winner-take-all" property. Figure 3(a) shows an example of initial networks with the number of actors $M=50$, the number of collaboration acts $N=40$, the average degree of actors $\langle m \rangle=3.0$, and the average degree of collaboration acts $\langle n \rangle=3.75$. The figure was

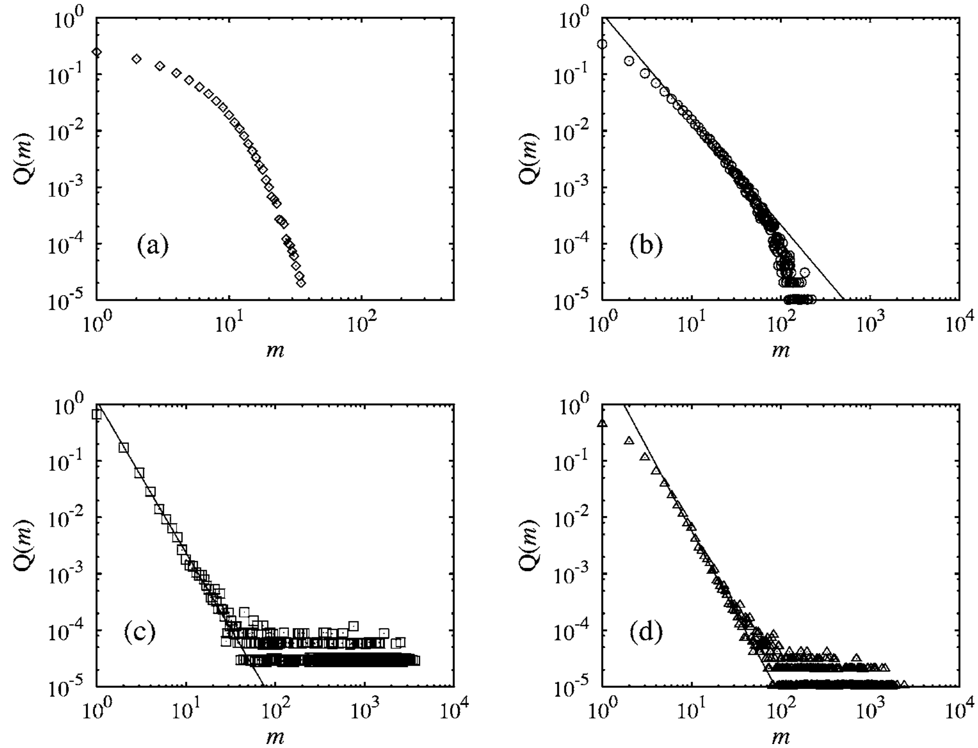

FIG. 4. The degree distributions of actors, $Q(m)$, in the case of (a) the delta-function fitness distribution, $\rho(\eta)=\delta(\eta-1)$; (b) the uniform fitness distribution, $\rho(\eta)=1$ $(0\leq\eta\leq 1)$; (c) the exponential fitness distribution, $\rho(\eta)=e^{-\eta}(0\leq\eta\leq+\infty)$; (d) the Poisson fitness distribution with the average 8. In the numerical experiments, we have used the initial network with $M=10000$, $N=8000$, $\langle m\rangle=3.0$, and $\langle n\rangle=3.75$. The number of the rewiring steps is $300M$, and the data are averaged over 20 realizations. In (b), (c), and (d), the solid lines correspond to $Q(m)\sim m^{-1.87}, m^{-2.70}$, and $m^{-2.94}$, respectively.

produced by using Pajek's software [24]. We have used the uniform fitness distribution $\rho(\eta)=1$ $(0\leq\eta\leq 1)$ and performed $100M$ steps of the preferential rewiring process for the initial networks; this is a snap shot after reaching a stationary state. We have confirmed that $100M$ steps are sufficiently long for obtaining stationary states in this case. Figure 3(b) shows the obtained network. As we can see in Fig. 3(b), there are a lot of isolated actors; there are 23 isolated actors in Fig. 3(b), while there is only one isolated actor in Fig. 3(a).

B. Degree distribution of actors

Figure 4 shows examples of the degree distribution of actors, $Q(m)$, of the obtained bipartite graphs for several fitness distribution by using the proposed preferential rewiring process. In the numerical experiments, we have used the initial networks with the number of actors $M=10000$, the number of collaboration acts $N=8000$, the average degree of actors $\langle m\rangle=3.0$, and the average degree of collaboration acts $\langle n\rangle=3.75$. We have applied $300M$ steps of the preferential rewiring process to the initial networks. We have checked the obtained degree distribution of actors and other properties from $200M$ steps to $500M$ steps, and found that those do not change. Then we can consider that a stationary state is already reached with $300M$ rewiring steps. The data are averaged over 20 realizations; we have performed the numerical experiments for 20 different initial networks with the same values of $M, N, \langle m\rangle$, and $\langle n\rangle$.

Here we use four different fitness distributions. First, we consider the delta-function fitness distribution given by $\rho(\eta)=\delta(\eta-1)$. In this case, all fitness parameters are the same values regardless of actors $(\eta_i=1, \forall i)$. Figure 4(a) shows the result in this case. The degree distribution $Q(m)$ does not show a power-law behavior; it has an exponential decay. Figure 4(b) shows the result in the second case in which the uniform fitness distribution given by $\rho(\eta)=1$ $(0\leq\eta\leq 1)$ is used. In this case, the network has a scale-free-like property; the meaning of *scale-free-like* is that a degree distribution has a power-law form for a wide range of m. The solid line in Fig. 4(b) corresponds to $Q(m)\sim m^{-1.87}$. Figures 4(c) and 4(d) show the results for the cases with the exponential fitness distribution given by $\rho(\eta)=e^{-\eta}$ $(0\leq\eta\leq+\infty)$ and the Poisson fitness distribution with the average 8, respectively. In both cases, while the degree distributions $Q(m)$ have the scale-free-like property except large m, it is possible that several actors have very large degrees because the degree distribution has a nondecaying tail for large m. These degree distributions suggest that parts of the whole network might be of starlike structure in which a lot of edges concentrate on an actor (or a few actors). In these cases, a finite fraction of edges may condense on a single actor (or a few actors). This phenomenon is called *condensation* of edges [25] (in Ref. [26], the phenomenon is called Bose-Einstein condensation). As for the condensation phenomenon, we per-

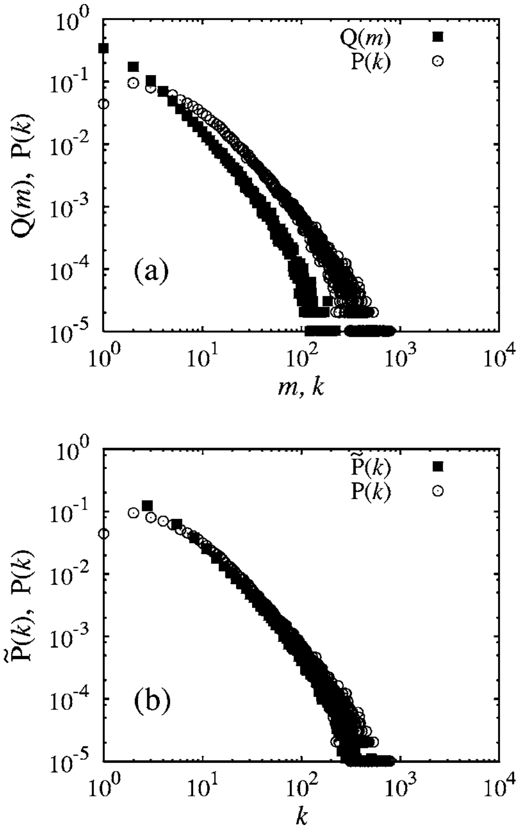

FIG. 5. (a) The degree distribution of actors, $Q(m)$, and that of the one-mode projection, $P(k)$. (b) The transformed degree distribution defined by Eq. (13), $\tilde{P}(k)$, and $P(k)$. These data were obtained by averaging over 20 realizations with the same parameter $M=10000$, $N=8000$, $\langle m\rangle=3.0$, and $\langle n\rangle=3.75$. Here we have used the uniform fitness distribution and $300M$ steps of the rewiring process.

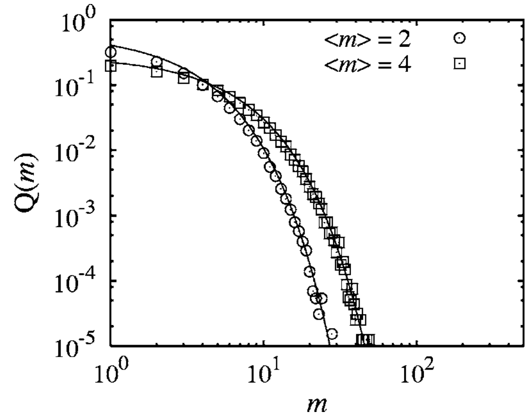

FIG. 6. The degree distribution of actors, $Q(m)$, in the bipartite graphs with $M=10000$ and $N=8000$ in the case of the delta-function fitness distribution. The number of the rewiring steps is $300M$ and the data are averaged over 20 realizations. The results for the average degree $\langle m\rangle=2.0$ ($\langle n\rangle=2.5$) are shown by circles and those for $\langle m\rangle=4.0$ ($\langle n\rangle=5.0$) by squares. The solid lines correspond to Eq. (16) for respective cases.

form more detailed study in Sec. V in order to clarify the characteristic nature of the fitness distribution responsible to the condensation of edges.

C. Degree distribution of one-mode projection

The scale-free property of a network composed of only one type of nodes or a one-mode projection of a bipartite graph is characterized by means of the degree distribution $P(k)$ with a fat tail, $P(k)\sim k^{-\gamma}$. In this subsection, we compare the degree distribution of actors in a bipartite graph and that of its one-mode projection. Ramasco *et al.* [4] have discussed that the power-law behavior of the degree distribution of actors, $Q(m)$, for the bipartite graph means that for its one-mode projection. We here confirm their discussion by numerical experiments.

An actor connected to a collaboration act with n edges has $n-1$ connections to another actors through the collaboration act. In the present paper, we have assumed that the degree distribution of collaboration acts, $S(n)$, is given by an exponential form, and hence we can consider that each collaboration act has approximately the same number of edges, $\langle n\rangle$. Therefore, we assume that the number of actors participating in each collaboration act is constant and equal to its average value $n=\langle n\rangle$. Within the above approximation, we have a following relation between the degree of an actor on a bipartite graph, m, and the degree on the one-mode projection, k:

$$k = m(\langle n\rangle - 1). \tag{12}$$

For large k and m, we consider both k and m to be continuous, so that we have

$$\tilde{P}(k) = \frac{1}{\langle n\rangle - 1} Q\left(\frac{k}{\langle n\rangle - 1}\right). \tag{13}$$

We expect that the transformed degree distribution $\tilde{P}(k)$ agrees with the degree distribution of the one-mode projection, $P(k)$. To confirm it, we have performed numerical experiments with the parameter $M=10000$, $N=8000$, $\langle m\rangle=3.0$, and $\langle n\rangle=3.75$. We set the fitness distribution as the uniform distribution, $\rho(\eta)=1$ ($0\leqslant\eta\leqslant1$), and repeat $300M$ rewiring processes. The data have been averaged over 20 realizations. Figure 5(a) shows the degree distribution of actors, $Q(m)$, and that of the one-mode projection, $P(k)$. They show similar behavior, but there is a clear difference between them. The transformed degree distribution of Eq. (13) is shown in Fig. 5(b). The degree distributions $\tilde{P}(k)$ and $P(k)$ are in good agreement. Therefore, we consider that Eq. (13) is confirmed by numerical experiments and the one-mode projection has the scale-free-like property when the degree distribution $Q(m)$ has the scale-free-like property for its corresponding bipartite graph.

D. Comparison between numerical results and analytical results

As mentioned in Sec. III B, we derive the degree distribution of actors, $Q(m)$, analytically for the case of the delta-function fitness distribution $\rho(\eta)=\delta(\eta-1)$. We make a com-

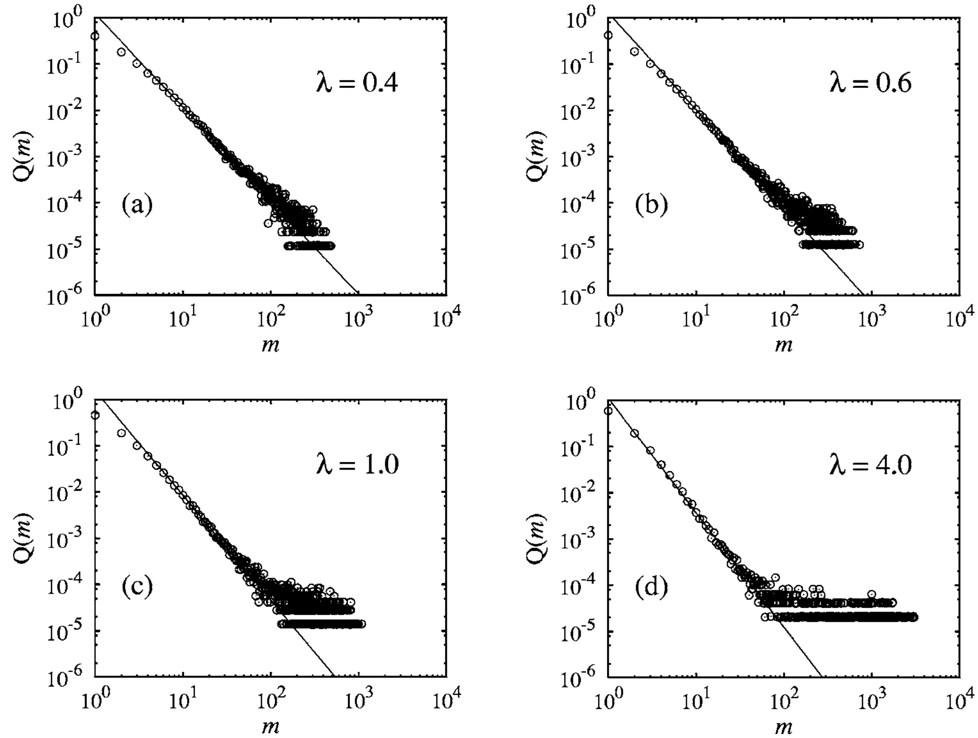

FIG. 7. The degree distribution of actors, $Q(m)$, in the case of the fitness distribution described by Eq. (17). We have used the following parameters: $M=10000$, $N=10000$, $\langle m\rangle=3.0$, and $\langle n\rangle=3.0$. (a) The case of $\lambda=0.4$. The solid line corresponds to $Q(m)\sim m^{-2.02}$. (b) The case of $\lambda=0.6$. The solid line corresponds to $Q(m)\sim m^{-2.10}$. (c) The case of $\lambda=1.0$. The solid line corresponds to $Q(m)\sim m^{-2.26}$. (d) The case of $\lambda=4.0$. The solid line corresponds to $Q(m)\sim m^{-2.47}$.

parison between the numerical results and the analytical results for this case.

Because all actors have the same fitness parameter 1, the factor Z is obtained as follows:

$$Z=\sum_m (m+1)q_m(1)=M\langle m\rangle+M, \tag{14}$$

so that $q_m(1)$ does not diverge because $Z>M\langle m\rangle$ in Eq. (9). Using Eqs. (9) and (10), we calculate the number of actors with m edges by

$$q_m=\int d\eta\,\rho(\eta)q_m(\eta)=\int d\eta\,\delta(\eta-1)q_0(\eta)\left(\frac{\eta M\langle m\rangle}{Z}\right)^m$$
$$=q_0(1)\left(\frac{M\langle m\rangle}{Z}\right)^m\sim e^{-m\ln(Z/M\langle m\rangle)}=e^{-m\ln(1+(1/\langle m\rangle))}. \tag{15}$$

Therefore, the degree distribution of actors is expressed as

$$Q(m)=\frac{q_m}{\sum_{m=1}^{\infty}q_m}\simeq\frac{q_m}{\int_1^{\infty}dk\,q_k}=\ln\left(1+\frac{1}{\langle m\rangle}\right)e^{(1-m)\ln(1+(1/\langle m\rangle))}. \tag{16}$$

In order to confirm the validity of the analytical treatment, we have performed numerical experiments. In the numerical experiments, we have used the initial networks with the sizes $M=10000$, $N=8000$, and calculated for two different cases of the average degrees; one is the case with $\langle m\rangle=2.0$ and $\langle n\rangle=2.5$, and the other with $\langle m\rangle=4.0$ and $\langle n\rangle=5.0$. We have performed $300M$ steps of the rewiring process and the data are averaged over 20 realizations. Figure 6 shows results of the numerical experiments and their corresponding analytical results. The solid lines in Fig. 6 correspond to Eq. (16) for respective cases. The analytical results are in good agreement with the results by the numerical experiments in both cases.

V. CONDENSATION OF EDGES

A. Investigations in systematic way

In this subsection, we give some detailed investigations on the condensation of edges mentioned in Sec. IV B. Bianconi and Barabási [27] have introduced the fitness parameters to their growing network model, and they have found that the fluctuation of the fitness parameters causes a phenomenon that a finite fraction of edges may condense on a single node with the highest fitness parameter. The model proposed by Bianconi and Barabási [27] is the growing one, but our model in the present paper is the non-growing one within a kind of adiabatic approximation. Therefore we think that the condensation phenomenon mentioned in Sec. IV B is not obvious, and the phenomenon should be studied in more detail.

We wish to study the condensation phenomenon in a systematic way. For this aim we use a one-parameter family of a fitness distribution and investigate its effect to the degree

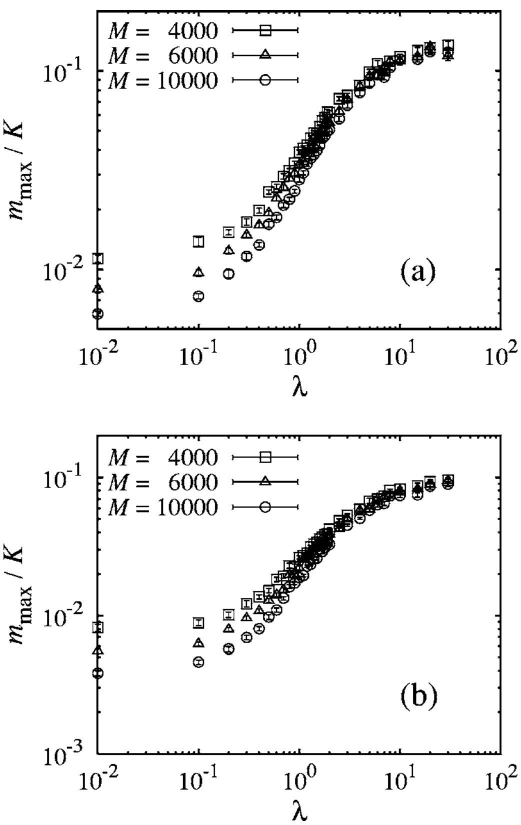

FIG. 8. Fractions of the total number of edges connected to the most connected actor, $m_{\max}/K$, plotted as a function of λ. (a) The fraction in the case of $\langle m \rangle = 3.0$ and $\langle n \rangle = 3.0$. We have calculated for different network sizes: $M=10000$ and $N=10000$; $M=6000$ and $N=6000$; $M=4000$ and $N=4000$. (b) The fraction in the case of $\langle m \rangle = 2.0$ and $\langle n \rangle = 4.0$. We have calculated for different network sizes: $M=10000$ and $N=5000$; $M=6000$ and $N=3000$; $M=4000$ and $N=2000$.

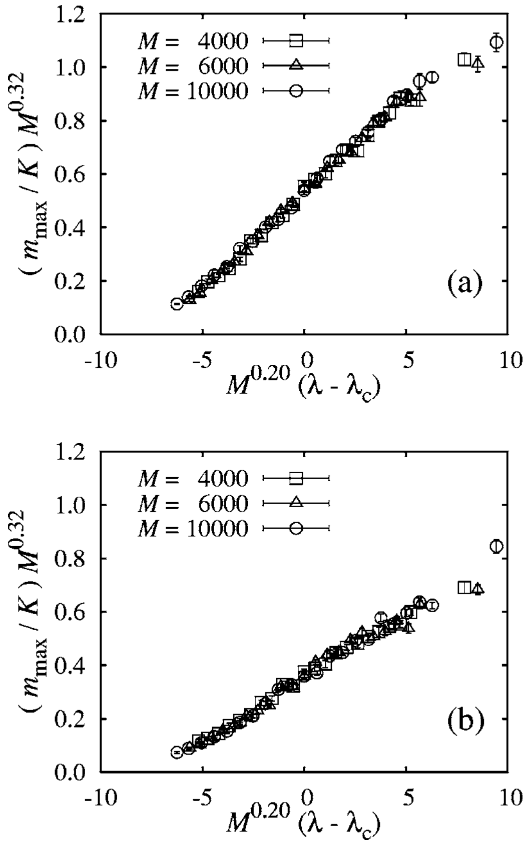

FIG. 9. Finite-size scaling plots of $m_{\max}/K$. (a) and (b) correspond to (a) and (b) in Fig. 8, respectively.

distribution. More definitely speaking, we use the following fitness distribution:

$$\rho(\eta) = (\lambda + 1)(1 - \eta)^{\lambda} \quad (0 \leqslant \eta \leqslant 1). \tag{17}$$

We control the form of the fitness distribution using the parameter λ. When the value of λ is small enough, the fitness distribution has the form similar to the uniform fitness distribution except near $\eta=1$. As increasing the value of λ, the fitness distribution changes its form gradually from the concave one $(0<\lambda<1)$ to the convex one $(\lambda>1)$.

First we have examined the degree distribution of actors, $Q(m)$. In numerical experiments, the following parameters have been used: $M=10000$, $N=10000$, $\langle m \rangle = 3.0$, and $\langle n \rangle = 3.0$. We have performed $300M$ steps of the preferential rewiring process, and averaged over 20 realizations. Figure 7 shows the results of the numerical experiments. The result obtained by using the parameter $\lambda=0.4$ is shown in Fig. 7(a). Figure 7(b) is the results with the parameter $\lambda=0.6$. We see clearly the occurrence of the condensation of edges in Figs. 7(c) and 7(d), in which the parameters λ are 1.0 and 4.0, respectively. In Fig. 7(a), the degree distribution does not have the nondecaying tail representing the condensation phenomenon, and the degree distribution is approximately fitted by using a straight line. However, the tail of the degree distribution gradually appears for large values of m as λ increases. In Figs. 7(c) and 7(d), the degree distributions have the nondecaying tails. Especially, actors with a very large degree, e.g., $m>1000$, emerge with the nonzero probability.

We have investigated the influence of the condensation on various properties of networks, such as a size of a giant cluster, a cluster coefficient of a one-mode projection. We have found that the condensation shows a clear effect on the ratio $m_{\max}/K$, where $m_{\max}$ is the degree of the most connected actor and K is the total number of edges in the bipartite graph. The value of $m_{\max}$ shows how much the most connected actor gets edges in the whole network, and then we consider that the value of $m_{\max}$ could become a measure of condensation.

Figures 8(a) and 8(b) show the fractions of the total number of edges connected to the most connected actor, $m_{\max}/K$, as a function of λ. In the numerical experiments, we have fixed the average degrees of actors and collaboration acts: $\langle m \rangle = 3.0$ and $\langle n \rangle = 3.0$ in Fig. 8(a), and $\langle m \rangle = 2.0$ and $\langle n \rangle = 4.0$ in Fig. 8(b). The numerical experiments for three different sizes are performed: $M=10000$ and $N=10000$, $M=6000$ and $N=6000$, $M=4000$ and $N=4000$ in Fig. 8(a), and $M=10000$ and $N=5000$, $M=6000$ and $N=3000$, $M=4000$ and $N=2000$ in Fig. 8(b). We see that the inflection points exist near $\lambda=1.0$ in Figs. 8(a) and 8(b), so that we infer that the condensation arises for $\lambda>1$. Figures 8(a) and 8(b) indi-

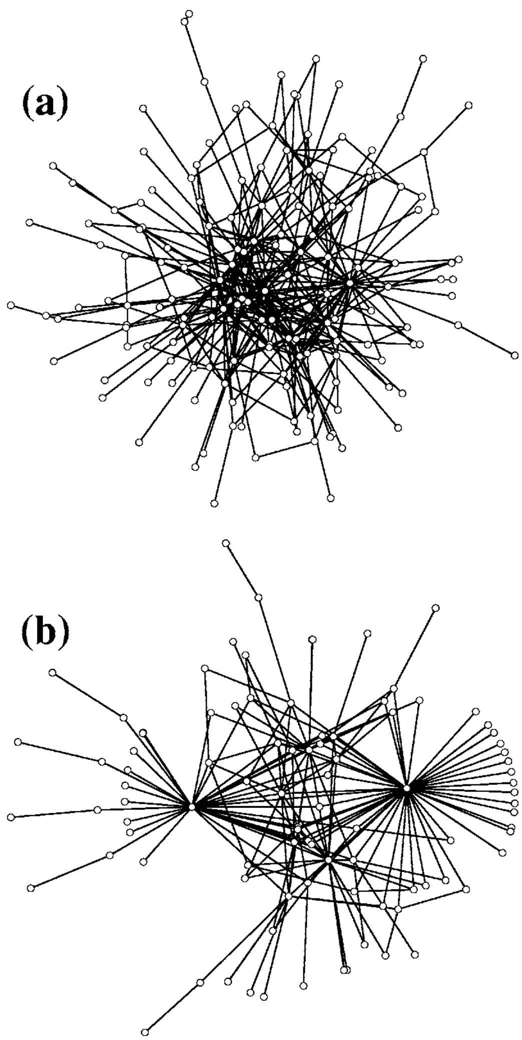

FIG. 10. Illustrations of the obtained giant cluster. (a) The case with the fitness distribution of Eq. (17) with $\lambda=0.4$. (b) The case with $\lambda=8.0$. In both cases, we have used the parameters of $M=400$, $N=400$, $\langle m\rangle=2.0$, and $\langle n\rangle=2.0$. After $100M$ steps of the rewiring process, we have made the one-mode projection of the obtained bipartite graph and removed isolated actors.

cate that the most connected actor maintains a finite fraction of the total number of edges even in the thermodynamic limit when the condensation of edges takes place (in the case of the large value of λ). In contrast, when the condensation does not occur, namely for small λ, the number of edges with the most connected actor gradually decreases with the increasing network size. This situation is similar to the one given by Bianconi and Barabási [26,27] in their growing model.

From Figs. 8(a) and 8(b), we infer that there is a threshold $\lambda_c=1.0$. We then apply the finite-size scaling [28] to the behavior of $m_{\max}/K$ near the threshold $\lambda_c=1.0$. We set the following finite-size scaling hypothesis near the threshold λ_c:

$$m_{\max}/K = M^{-\beta/\nu}F(M^{1/\nu}(\lambda-\lambda_c)), \qquad (18)$$

where $F(x)$ is a scaling function, β and ν are critical exponents. In Figs. 9(a) and 9(b), we plot $(m_{\max}/K)M^{\beta/\nu}$ as a function of $M^{1/\nu}(\lambda-\lambda_c)$. The parameters for numerical experiments in Figs. 9(a) and 9(b) are the same as in Figs. 8(a) and 8(b), respectively. We here have chosen the critical exponents as $\beta=1.6$ and $\nu=5.0$. Although we have used the different parameters of the average degrees $\langle m\rangle$ and $\langle n\rangle$ in Figs. 9(a) and 9(b), the common critical exponents give good scaling results in both cases. Therefore, we expect that there is a certain universality in the condensation phenomenon though more detailed numerical experiments and discussion would be needed for further study of the critical phenomenon.

From these results, we consider that the condensation of edges arises when there are a relatively small number of actors with large fitness parameters. This agrees with the fact that the condensation phenomenon takes place in the cases of the exponential fitness distribution and the Poisson fitness distribution, as mentioned in Sec. IV A. The condensation phenomenon could be discussed in terms of the zero-range process [22,29], and a related model of nongrowing networks has been proposed [30]. However, most of these works have focused their attention in homogeneous cases for fitness parameters. Generally speaking, the heterogeneous case such as the model in the present paper, in which fitness parameters are different from each other, is difficult to be analyzed [31]. Although we have not succeeded in the analytical treatments for the condensation phenomenon in our model yet, the critical exponents obtained by the finite size scaling would be important to study the condensation phenomenon from a viewpoint of statistical physics.

B. Illustration of networks

Finally, we give examples of network structure obtained by numerical experiments for the fitness distribution given in Eq. (17). We have set the initial parameters as $M=400$, $N=400$, $\langle m\rangle=2.0$, and $\langle n\rangle=2.0$ for the numerical experiments, and here we use $\lambda=0.4$ and 8.0 as the parameter of the fitness distributions. In these cases, we have checked that a stationary state is reached with $100M$ rewiring steps. After $100M$ steps of the preferential rewiring process, we have made the one-mode projection from the obtained bipartite graph and removed isolated actors from the one-mode projection. Figures 10(a) and 10(b) show the one-mode projections without isolated actors. In Figs. 10(a) and 10(b), each edge is depicted as a single edge even if the edge is multiple. There are 165 actors in Fig. 10(a), and 108 actors in Fig. 10(b). In Fig. 10(b), two actors get many more edges than the other actors and these actors make a starlike structure in the network. This starlike structure corresponds to the condensation phenomenon.

VI. CONCLUSION

We have proposed a model for generation of complex bipartite graphs. This model is based on the preferential attachment mechanism with a fitness distribution function, but a bipartite graph in our model is not growing: the number of collaboration acts, that of actors, and that of edges are fixed. This nongrowing model is considered to be an adiabatic approximation for collaboration networks in which the size of networks does not change rapidly, but the structure of the networks change with time rapidly. The preferential attachment mechanism alone does not give networks with the scale-free property in the nongrowing case. We have shown that one more factor is necessary in order to generate networks with the scale-free property. The factor is the fitness parameter, which corresponds to the propensity of nodes to gain edges. We have performed various numerical experiments in which the preferential rewiring process is applied to bipartite graphs, and demonstrated that some obtained networks have the scale-free-like property by using suitable fitness distributions. It has been shown that a non-fat-tailed fitness distribution can generate a network with a fat-tailed degree distribution. We have also given analytical results in a special case in which the degree distribution is derived explicitly. Furthermore, it has been clarified that the condensation of edges takes place when there are a small number of actors with large fitness parameters. We have estimated the critical exponents by using the finite-size scaling.

We believe that nongrowing models are important in order to understand various complex networks, because the nongrowing models are considered as a kind of adiabatic approximation of some real networks. For this reason, we have investigated the nongrowing model for generation of complex bipartite graphs in the present paper. As a future work, the other degree distribution of collaboration acts, $S(n)$, is also interesting and should be investigated (for example, the Gaussian distribution or the Poisson distribution). Furthermore, we are also interested in a relationship between the nongrowing model and the zero-range process.

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